

PROCEEDINGS OF THE ROYAL SOCIETY.

SECTION A.—MATHEMATICAL AND PHYSICAL SCIENCES.

Zonal Harmonics of High Order in Terms of Bessel Functions.

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(Communicated by Prof. J. W. Nicholson, F.R.S. Received January 13, 1919.)

The zonal harmonic $P_n(\theta)$, for large values of n and small values of θ , is approximately given by the formula

$$P_n(\theta) = J_0(a\theta),$$

where $a^2 = n(n+1)$.

This result is easily derived from a consideration of Legendre's equation,

$$\frac{d^2u}{d\theta^2} + \cot \theta \cdot \frac{du}{d\theta} + n(n+1)u = 0, \quad u = P_n(\theta). \quad (1)$$

The first approximation of $\cot \theta$ is $1/\theta$, and substituting this value in (1), the equation becomes

$$\frac{d^2u}{d\theta^2} + \frac{1}{\theta} \cdot \frac{du}{d\theta} + n(n+1)u = 0, \quad (2)$$

of which a solution is $J_0(a\theta)$, where $a^2 = n(n+1)$, a result similar to the one discovered by Mehler,*

$$\text{Lt}_{n \rightarrow \infty} P_n \left(\cos \frac{x}{n} \right) = J_0(x). \quad (3)$$

The second solutions of (2), Bessel functions of the second kind or Neumann functions, do not appear in the expression for $P_n(\theta)$.

Macdonald† has given two formulæ for $P_n(\theta)$ in terms of Bessel functions.

If $x = (2n+1) \sin \frac{\theta}{2}$,

$$P_n(\theta) = J_0(x), \quad (4)$$

* F. G. Mehler, 'Math. Annalen,' vol. 5, pp. 140-144 (1872).

† H. M. Macdonald, 'Proc. London Math. Soc.,' vol. 31 (Old Series), p. 269 (1899); *ibid.*, vol. 13 (New Series), pp. 220-1 (1914).

and

$$\begin{aligned} P_n(\theta) = J_0(x) + \sin^2 \frac{\theta}{2} \left\{ \frac{x}{6} J_3(x) - J_2(x) + \frac{1}{2x} J_1(x) \right\} \\ + \sin^4 \frac{\theta}{2} \left\{ \frac{x^2}{72} J_6(x) - \frac{11x}{30} J_5(x) + \frac{31}{12} J_4(x) - \frac{29}{6x} J_3(x) + \frac{9}{8x^2} J_2(x) \right\} \\ + \dots\dots\dots, \end{aligned} \quad (5)$$

whilst Lord Rayleigh* obtained the result

$$P_n(\theta) = \left(1 + \frac{z^2}{12a^2}\right) J_0(z) + \frac{z}{6a^2} J_0'(z), \quad (6)$$

in which $z = a\theta$ and $a^2 = n(n+1)$.

Further terms in the coefficients of $J_0(z)$ and $J_0'(z)$ can be found without difficulty by successive approximations. Equation (6) may be written

$$u = P_n(\theta) = \left(1 + \frac{\theta^2}{12}\right) J_0(a\theta) + \frac{\theta}{6a} J_0'(a\theta). \quad (7)$$

Consequently
$$\frac{du}{d\theta} = a \left(1 + \frac{\theta^2}{12}\right) J_0'(a\theta), \quad (8)$$

and since
$$\cot \theta = \frac{1}{\theta} - \frac{\theta}{3} - \frac{\theta^3}{45} - \dots,$$

equation (1) becomes

$$\begin{aligned} \frac{d^2u}{d\theta^2} + \frac{1}{\theta} \cdot \frac{du}{d\theta} + a^2\theta = a \left(\frac{\theta}{3} + \frac{\theta^3}{45}\right) \left(1 + \frac{\theta^2}{12}\right) J_0'(a\theta) \\ = a \left(\frac{\theta}{3} + \frac{\theta^3}{20}\right) J_0'(a\theta). \end{aligned} \quad (9)$$

Therefore
$$\frac{d^2u}{dz^2} + \frac{1}{z} \cdot \frac{du}{dz} + u = \frac{1}{a^2} \left(\frac{z}{3} + \frac{z^3}{20a^2}\right) J_0'(z), \quad (10)$$

or, the dependent variable u being changed to $vJ_0(z)$,

$$\frac{d^2v}{dz^2} + \frac{dv}{dz} \left(\frac{2J_0'}{J_0} + \frac{1}{z}\right) = \frac{1}{a^2} \left(\frac{z}{3} + \frac{z^3}{20a^2}\right) \frac{J_0'}{J_0}, \quad (11)$$

where the argument of the J functions is z .

Integrating (11), we find

$$\frac{dv}{dz} = \frac{1}{zJ_0^2} \left\{ A + \frac{1}{3a^2} \int z^2 J_0 J_0' dz + \frac{1}{20a^4} \int z^4 J_0 J_0' dz \right\}. \quad (12)$$

Now
$$\int z^2 J_0 J_0' dz = -\frac{z^2}{2} J_1^2, \quad \int z^4 J_0 J_0' dz = -\frac{z^4}{6} (2J_1^2 - J_2^2).$$

It is easily seen that

$$\int z^4 J_0 J_1 dz = \int z^2 \cdot z^2 J_0 J_1 dz = \frac{z^4 J_1^2}{2} - \int z^3 J_1^2 dz.$$

* Lord Rayleigh, 'Roy. Soc. Proc.', 1916, pp. 433-6.

But
$$\int z^3 J_1^2 dz = \frac{z^4 J_1^2}{4} - \frac{1}{2} \int z^4 J_1 \left(\frac{J_1}{z} - J_2 \right) dz,$$

and therefore
$$\int z^3 J_1^2 dz = \frac{z^4 J_1^2}{6} + \frac{1}{3} \int z^4 J_1 J_2 dz$$

$$\int z^4 J_1 J_2 dz = \int z^3 J_2 \cdot z J_1 dz = \int z^3 J_2 (z J_2' + 2 J_2) dz$$

$$= \int (z^4 J_2 J_2' + 2 z^3 J_2^2) dz = \frac{1}{2} \int d. z^4 J_2^2 = \frac{z^4 J_2^2}{2}.$$

Thus
$$\int z^4 J_0 J_1 dz = \frac{z^4 J_1^2}{2} - \frac{z^4 J_1^2}{6} - \frac{1}{3} \cdot \frac{z^4 J_2^2}{2} = \frac{z^4}{6} (2 J_1^2 - J_2^2) \quad (13)$$

Hence
$$\frac{dv}{dz} = \frac{1}{z J_0^2} \left\{ A - \frac{z^2}{6 a^2} J_1^2 - \frac{z^4}{120 a^4} (2 J_1^2 - J_2^2) \right\},$$

$$v = B + A \int \frac{dz}{z J_0^2} - \frac{1}{6 a^2} \int \frac{z J_1^2}{J_0^2} dz - \frac{1}{120 a^4} \int \frac{z^3 (2 J_1^2 - J_2^2)}{J_0^2} dz. \quad (14)$$

From the recurrence formula, J_2^2 may be expressed in terms of J_0 and J_1 ; the integrals now occurring on the right-hand side of (14) are $\int \frac{z J_1^2}{J_0^2} dz$ and $\int \frac{z^3 J_1^2}{J_0^2} dz + 2 \int \frac{z^2 J_1}{J_0} dz$.

Integrating $\int \frac{z^3 J_1^2}{J_0^2} dz$ by parts, we get

$$\int \frac{z^3 J_1^2}{J_0^2} dz = \int \frac{z^2 \cdot z J_1^2}{J_0^2} dz = z^2 \left(\frac{z J_1}{J_0} - \frac{z^2}{2} \right) - 2 \int z \left(\frac{z J_1}{J_0} - \frac{z^2}{2} \right) dz$$

$$= \frac{z^3 J_1}{J_0} - \frac{z^4}{2} - 2 \int \frac{z^2 J_1}{J_0} dz + \frac{z^4}{4}, \quad (15)$$

and therefore
$$\int \frac{z^3 J_1^2}{J_0^2} dz + 2 \int \frac{z^2 J_1}{J_0} dz = \frac{z^3 J_1}{J_0} - \frac{z^4}{4}. \quad (16)$$

Since $B = 1$ and $A = 0$, we find that

$$v = 1 - \frac{1}{6 a^2} \left\{ \left(z - \frac{z}{5 a^2} + \frac{z^3}{10 a^2} \right) \frac{J_1}{J_0} - \left(\frac{z^2}{2} - \frac{z^2}{10 a^2} + \frac{3 z^4}{80 a^2} \right) \right\}, \quad (17)$$

and
$$u = \left(1 + \frac{z^2}{12 a^2} + \frac{z^4}{160 a^4} + \dots - \frac{z^2}{60 a^4} \dots \right) J_0(z)$$

$$- \left(\frac{z}{6 a^2} + \frac{z^3}{60 a^4} + \dots - \frac{z}{30 a^4} \dots \right) J_1(z)$$

$$= \left(1 + \frac{\theta^2}{12} + \frac{\theta^4}{160} + \dots - \frac{\theta^2}{60 a^2} \dots \right) J_0(a\theta)$$

$$+ \frac{\theta}{6 a} \left(1 + \frac{\theta^2}{10} + \dots - \frac{1}{5 a^2} \dots \right) J_0'(a\theta). \quad (18)$$

For the next approximation we obtain from (18) the value of $du/d\theta$ as before. In this case

$$\frac{du}{d\theta} = \left\{ a \left(1 + \frac{\theta^2}{12} + \frac{\theta^4}{160} \dots \right) + \frac{\theta^2}{60a} \right\} J_0' + \frac{\theta^3}{120} J_0.$$

When this is substituted, we find eventually the value of dv/dz , which now contains the term $\int z^5 J_0^2 dz$ as well as integrals having the product $J_0 J_0'$ in the integrand. But the former can be expressed in terms of the latter. Thus, integrating by parts,

$$\int z^5 J_0 J_0' dz = \frac{z^6 J_0^2}{2} - 3 \int z^5 J_0^2 dz,$$

$$\text{or, generally,} \quad \int z^{2m} J_0 J_0' dz = \frac{z^{2m+1} J_0^2}{2} - m \int z^{2m-1} J_0^2 dz. \quad (19)$$

Consequently, the problem of finding dv/dz depends upon the determination of the integral $\int z^{2m} J_0 J_1 dz$.

Now

$$\begin{aligned} \int z^{2m} J_{m-1} J_m dz &= \int z^{2m-1} J_m \cdot z J_{m-1} dz \\ &= \int z^{2m-1} J_m (z J_m' + m J_m) dz = \frac{z^{2m} J_m^2}{2}. \end{aligned} \quad (20)$$

$$\begin{aligned} \text{Also} \quad \int z^{2m} J_{p-1} J_p dz &= \int z^{2m-2p} z^{2p} J_{p-1} J_p dz \\ &= \frac{z^{2m}}{2} J_p^2 - (m-p) \int z^{2m-1} J_p^2 dz. \end{aligned} \quad (21)$$

$$\begin{aligned} \text{But} \quad \int z^{2m-1} J_p^2 dz &= \frac{z^{2m}}{2m} J_p^2 - \frac{1}{m} \int z^{2m} J_p J_p' dz \\ &= \frac{z^{2m}}{2m} J_p^2 - \frac{1}{m} \int z^{2m} J_p \left(\frac{p}{z} J_p - J_{p+1} \right) dz \\ &= \frac{z^{2m}}{2m} J_p^2 - \frac{p}{m} \int z^{2m-1} J_p^2 dz + \frac{1}{m} \int z^{2m} J_p J_{p+1} dz, \end{aligned}$$

and therefore

$$(m+p) \int z^{2m-1} J_p^2 dz = \frac{z^{2m}}{2} J_p^2 + \int z^{2m} J_p J_{p+1} dz, \quad (22)$$

or, combining the results in (21) and (22),

$$\begin{aligned} \int z^{2m} J_{p-1} J_p dz &= \frac{z^{2m}}{2} J_p^2 - \frac{(m-p)}{(m+p)} \left\{ \frac{z^{2m} J_p^2}{2} + \int z^{2m} J_p J_{p+1} dz \right\} \\ &= \frac{p}{m+p} \frac{z^{2m} J_p^2}{2} - \frac{m-p}{m+p} \int z^{2m} J_p J_{p+1} dz. \end{aligned} \quad ($$

Applying this formula, making $p = 1, 2, 3, \dots$, in succession, we obtain

$$\begin{aligned} \int z^{2m} J_0 J_1 dz &= \frac{z^{2m}}{2} J_1^2 - \frac{(m-1)}{(m+1)} \left[\frac{z^{2m}}{2} J_1^2 + \int z^{2m} J_1 J_2 dz \right] \\ &= \frac{z^{2m}}{2} J_1^2 - \frac{(m-1)}{(m+1)} \left[\frac{z^{2m}}{2} J_1^2 + \frac{z^{2m}}{2} J_2^2 \right. \\ &\quad \left. - \frac{(m-2)}{(m+2)} \left\{ \frac{z^{2m}}{2} J_2^2 + \int z^{2m} J_2 J_3 dz \right\} \right]. \quad (24) \end{aligned}$$

Finally, we reach the term $\int z^{2m} J_{m-1} J_m dz$. Thus

$$\begin{aligned} \int z^{2m} J_0 J_1 dz &= z^{2m} \left[\frac{J_1^2}{m+1} - \frac{2(m-1)}{(m+1)(m+2)} J_2^2 + \frac{3(m-1)(m-2)}{(m+1)(m+2)(m+3)} J_3^2 - \dots \right] \\ &= \frac{z^{2m}}{m+1} \left[J_1^2 + \sum_{r=1}^{r=m-1} \cos r\pi \frac{(r+1)(m-1)(m-2) \dots (m-r)}{(m+2)(m+3) \dots (m+r+1)} J_{r+1}^2 \right]. \quad (25) \end{aligned}$$

Similarly,

$$\begin{aligned} \int z^{2m} J_{p-1} J_p dz &= \frac{z^{2m}}{m+p} \left[p J_p^2 + \sum_{r=1}^{r=m-p} \cos r\pi \right. \\ &\quad \left. \frac{(r+p)(m-p)(m-p-1) \dots (m-p-r+1)}{(m+p+1)(m+p+2) \dots (m+p+r)} J_{r+p}^2 \right]. \quad (26) \end{aligned}$$

$$\begin{aligned} \int z^{2m-1} J_p^2 dz &= \frac{z^{2m}}{m+p} \left[\frac{J_p^2}{2} - \frac{1}{m-p} \sum_{r=1}^{r=m-p} \cos r\pi \right. \\ &\quad \left. \frac{(r+p)(m-p)(m-p-1) \dots (m-p-r+1)}{(m+p+1)(m+p+2) \dots (m+p+r)} J_{r+p}^2 \right]. \quad (27) \end{aligned}$$

In the final step of each approximation, to obtain the value of v it is necessary to evaluate integrals of the type $\int \frac{z^{2m+1} J_1^2}{J_0^2} dz$ and $\int \frac{z^{2m} J_1}{J_0} dz$. These integrals occur under the form

$$\int \frac{z^{2m+1} J_1^2}{J_0^2} dz + 2m \int \frac{z^{2m} J_1}{J_0} dz. \quad m = 0, 1, 2, \dots$$

Integrating the first of these integrals by parts, and knowing that

$$\int \frac{z J_1^2}{J_0^2} dz = \frac{z J_1}{J_0} - \frac{z^2}{2},$$

we obtain the result

$$\int \frac{z^{2m+1} J_1^2}{J_0^2} dz = \frac{z^{2m+1} J_1}{J_0} - \frac{z^{2m+2}}{2m+2} - 2m \int \frac{z^{2m} J_1}{J_0} dz. \quad (28)$$

After further approximations, the expression for $P_n(\theta)$ in terms of $J_0(a\theta)$ and $J_1(a\theta)$ is as follows:—

$$\begin{aligned} P_n(\theta) &= \left(\alpha_0 - \frac{\beta_0}{a^2} + \frac{\gamma_0}{a^4} - \frac{\delta_0}{a^6} + \dots \right) J_0(a\theta) \\ &\quad - \frac{\theta}{6a} \left(\alpha_1 - \frac{\beta_1}{a^2} + \frac{\gamma_1}{a^4} - \frac{\delta_1}{a^6} + \dots \right) J_1(a\theta), \quad (29) \end{aligned}$$

where $\alpha_0 = 1 + \frac{\theta^2}{12} + \frac{\theta^4}{160} + \frac{61\theta^6}{120960} + \frac{1261\theta^8}{29030400} + \frac{79\theta^{10}}{20275000} + \dots,$

$$\beta_0 = \frac{\theta^2}{60} + \frac{\theta^4}{420} + \frac{19\theta^6}{60480} + \dots,$$

$$\gamma_0 = \frac{2\theta^2}{315} + \frac{\theta^4}{560} + \dots, \quad \delta_0 = \frac{\theta^2}{210} + \dots,$$

$$\alpha_1 = 1 + \frac{\theta^2}{10} + \frac{31\theta^4}{3360} + \frac{17\theta^6}{18900} + \frac{9689\theta^8}{106444800} + \dots,$$

$$\beta_1 = \frac{1}{5} + \frac{4\theta^2}{105} + \frac{31\theta^4}{5040} + \dots,$$

$$\gamma_1 = \frac{8}{105} + \frac{\theta^2}{35} + \dots, \quad \delta_1 = \frac{2}{35} + \dots.$$

To calculate $P_n(\theta)$ to six places of decimals, when n is equal to or greater than 20, only α_0 , β_0 , α_1 , and β_1 need to be tabulated. The following tables have been constructed, giving these coefficients for each angle of the quadrant:—

θ° .	α_0 .	β_0 .	α_1 .	β_1 .
0	1·0000000	0·00000	1·00000	0·2000
1	1·0000254	0·00000	1·00003	0·2000
2	1·0001016	0·00002	1·00012	0·2001
3	1·0002235	0·00004	1·00027	0·2001
4	1·0004063	0·00008	1·00049	0·2002
5	1·0006350	0·00012	1·00076	0·2003
6	1·0009146	0·00018	1·00110	0·2004
7	1·0012453	0·00024	1·00149	0·2006
8	1·0016270	0·00032	1·00195	0·2008
9	1·0020600	0·00041	1·00247	0·2010
10	1·0025443	0·00051	1·00305	0·2012
11	1·0030801	0·00062	1·00369	0·2014
12	1·0036675	0·00074	1·00440	0·2017
13	1·0043067	0·00086	1·00517	0·2020
14	1·0049978	0·00100	1·00601	0·2023
15	1·0057411	0·00115	1·00690	0·2026
16	1·0065368	0·00131	1·00786	0·2030
17	1·0073850	0·00148	1·00888	0·2034
18	1·0082860	0·00166	1·00996	0·2038
19	1·0092401	0·00186	1·01111	0·2042
20	1·0102476	0·00206	1·01232	0·2047
21	1·0113087	0·00228	1·01360	0·2052
22	1·0124237	0·00250	1·01495	0·2057
23	1·0135930	0·00275	1·01636	0·2063
24	1·0148168	0·00300	1·01784	0·2069
25	1·0160956	0·00326	1·01938	0·2075
26	1·0174297	0·00353	1·02099	0·2081
27	1·0188194	0·00382	1·02267	0·2088
28	1·0202652	0·00412	1·02442	0·2095
29	1·0217675	0·00443	1·02624	0·2102
30	1·0233267	0·00475	1·02813	0·2109
31	1·0249433	0·00509	1·03009	0·2117
32	1·0266179	0·00544	1·03212	0·2125
33	1·0283508	0·00580	1·03423	0·2133

θ°	α_0	β_0	α_1	β_1
34	1·0301425	0·00618	1·03640	0·2142
35	1·0319937	0·00657	1·03865	0·2151
36	1·0339049	0·00697	1·04097	0·2160
37	1·0358766	0·00739	1·04337	0·2170
38	1·0379095	0·00782	1·04585	0·2180
39	1·0400042	0·00827	1·04841	0·2191
40	1·0421613	0·00873	1·05104	0·2201
41	1·0443815	0·00920	1·05375	0·2212
42	1·0466655	0·00969	1·05655	0·2224
43	1·0490140	0·01020	1·05942	0·2236
44	1·0514277	0·01073	1·06238	0·2248
45	1·0539074	0·01127	1·06542	0·2261
46	1·0564539	0·01183	1·06855	0·2274
47	1·0590681	0·01240	1·07176	0·2287
48	1·0617507	0·01299	1·07506	0·2301
49	1·0645027	0·01360	1·07845	0·2316
50	1·0673250	0·01423	1·08193	0·2330
51	1·0702185	0·01488	1·08551	0·2346
52	1·0731842	0·01554	1·08918	0·2361
53	1·0762231	0·01623	1·09294	0·2377
54	1·0793362	0·01693	1·09680	0·2394
55	1·0825246	0·01766	1·10076	0·2411
56	1·0857894	0·01841	1·10482	0·2429
57	1·0891317	0·01918	1·10898	0·2448
58	1·0925528	0·01997	1·11324	0·2466
59	1·0960538	0·02078	1·11761	0·2486
60	1·0996361	0·02162	1·12209	0·2506
61	1·1033009	0·02248	1·12668	0·2526
62	1·1070496	0·02337	1·13138	0·2548
63	1·1108835	0·02428	1·13620	0·2569
64	1·1148042	0·02522	1·14114	0·2592
65	1·1188132	0·02619	1·14619	0·2615
66	1·1229117	0·02719	1·15136	0·2639
67	1·1271016	0·02821	1·15667	0·2664
68	1·1313845	0·02927	1·16209	0·2690
69	1·1357621	0·03036	1·16765	0·2716
70	1·1402361	0·03147	1·17334	0·2744
71	1·144808	0·0326	1·1792	0·277
72	1·149481	0·0338	1·1851	0·280
73	1·154256	0·0350	1·1913	0·283
74	1·159134	0·0363	1·1975	0·286
75	1·164119	0·0376	1·2039	0·289
76	1·169212	0·0389	1·2105	0·293
77	1·174416	0·0403	1·2172	0·296
78	1·179733	0·0417	1·2241	0·300
79	1·185165	0·0432	1·2312	0·303
80	1·190715	0·0447	1·2384	0·307
81	1·196386	0·0462	1·2457	0·311
82	1·202179	0·0478	1·2533	0·315
83	1·208098	0·0494	1·2610	0·319
84	1·214147	0·0511	1·2690	0·324
85	1·220327	0·0529	1·2771	0·328
86	1·226641	0·0547	1·2855	0·333
87	1·233094	0·0565	1·2940	0·338
88	1·239688	0·0584	1·3028	0·342
89	1·246427	0·0604	1·3117	0·348
90	1·253314	0·0625	1·3209	0·352

The formulæ and tables are of course especially useful when θ is a small angle. They have been tested by computing $P_n(\theta)$ for comparatively small

values of n and large values of θ . Even in these extreme cases the results are correct to six places of decimals.

For the purposes of comparison, a short table of $P_8(\theta)$ has been computed, giving in the second column the values of this function when the angle θ is large, both γ_0 and γ_1 being required in these calculations. In the third column are the values calculated by Tallquist.*

$P_8(\theta)$.		
θ .	Calculated from (29).	Tallquist's values.
60°	-0.073638 (9)	-0.0736389
65°	-0.241143 (9)	-0.2411438
70°	-0.278015 (2)	-0.2780152
75°	-0.170219 (8)	-0.1702200
80°	+0.023307 (8)	+0.0233079
85°	+0.201746 (2)	+0.2017462
90°	+0.273437 (6)	+0.2734375

In the construction of this table $J_0(a\theta)$ and $J_1(a\theta)$ were calculated from Meissel's tables. When $a\theta$ exceeds 15.5, J_0 and J_1 can be found to seven places of decimals from formulæ† derived from the addition theorem of these functions.

* Tallquist, 'Acta Soc. Sc. Fennicæ' (Helsingfors), vol. 33, No. 4, pp. 1-8. Tables of $P_n(\cos \theta)$ for $n = 1, 2, 3 \dots 8$ to ten places of decimals for each degree of the quadrant; A. Lodge, Appendix to Lord Rayleigh's paper, "On the Acoustic Shadow of a Sphere," Tables of Zonal Harmonics for 5° intervals in the value of θ from 0° to 90°, and $n = 1, 2, 3 \dots 20$ to seven places of decimals, 'Phil. Trans.,' A, vol. 203, pp. 87-110 (1904).

† "The Addition Theorem of Bessel Functions of Zero and Unit Orders," 'Phil. Mag.,' vol. 36, September, 1918.